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Abstract

We revisit two-person one-dimensional pure location games à la Anderson et al. (1992) and show that they admit continuous best-response potential functions (Voorneveld, 2000) if demand is sufficiently elastic (to the extent that the Principle of Minimum Differentiation fails); if demand is not that elastic (or is completely inelastic) they still admit continuous quasi-potential functions (Schipper, 2004). We also show that, even if a continuous best-response potential function exists, a generalized ordinal potential function (Monderer and Shapley, 1996) need not exist.

Keywords: symmetric games, location games, best-response potential games, pure Nash equilibrium existence

JEL Classification: C72 (Noncooperative game)

1 Introduction

Location games date back to Hotelling (1929), originally as a two-person game of location and price choice by sellers in a market depicted by a line segment. Hotelling assumed that buyers are uniformly distributed, and each buyer always demands a unit quantity of the product from a seller whose delivered price is cheaper. Thus, the buyers are assumed to have completely inelastic demand. Smithies (1941) was the first who considered a situation where demand of buyers is strictly decreasing in delivered price, i.e., a situation where buyers have elastic demand. Here, the delivered price is decomposed into the mill price and transportation cost that increases with distance. By letting mill price and per distance

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transportation cost be equal across sellers, we have a pure location game. A thorough study of such pure location games is done by Anderson et al. (1992, Chapter 8.2). They showed, in particular, that if demand is completely inelastic, or elastic but not too elastic, the two sellers agglomerate at the center of the market at equilibrium, namely, Hotelling's *Principle of Minimum Differentiation* holds; if, on the other hand, the demand is sufficiently elastic, the central agglomeration at equilibrium disappears, i.e., the principle ceases to hold true.[1] See also Gabszewicz and Thisse (1992) for pure location games in general.

In this paper, we revisit two-person pure location games à la Anderson et al. (1992),[2] shedding some new light on them. We show that these games admit continuous *best-response potential functions* (Voorneveld, 2000) if demand is sufficiently decreasing in distance[3] (to the extent that the central agglomeration ceases to hold true); if demand is not that decreasing (or is constant) they still admit continuous *quasi-potential functions* (Schipper, 2004). Thus, in these games, continuous potential functions are securing the existence of a pure Nash equilibrium. We also show that even if a continuous best-response potential function exists, a *generalized ordinal potential function* (Monderer and Shapley, 1996) need not exist.

The rest of the paper is organized as follows. In Section 2, we define the game and give some preliminaries. In Section 3, we show our main results. Some concluding remarks are given in Section 4.

2 Preliminaries

2.1 The game

Denote by S the compact real interval $[0, L]$, where L is a positive real number. Also, denote by f a real-valued continuous positive function defined on $[0, \infty[$ that is constant or strictly decreasing. Throughout, f is assumed to be continuous and positive, unless otherwise stated. We denote by $G = (S, f)$ a two-person game in strategic form such that the strategy set is S and the payoff functions $u_i \colon S \times S \to \mathbb{R}$ $(i = 1, 2)$ are defined as follows:

$$u_i(x_1, x_2) := \int_{V_i(x_1,x_2)} f(|y - x_i|)dy + \frac{1}{2}\int_{V_0(x_1,x_2)} f(|y - x_i|)dy \tag{1}$$

[1] We note that their condition of agglomeration (Anderson et al., 1992, page 282, Eq. (6)) is equivalent to (the converse of) our Eq. (8) that demarcates our two cases.

[2] To be fair, we note that their analysis of two-person pure location games is just an opening of their comprehensive analyses of location games with more than two players, entry, and price competition; our analysis is limited in scope compared to theirs.

[3] The decreasingness of demand in delivered price turns into the decreasingness of demand in distance in pure location games.

where, with $\{i,j\} = \{1,2\}$, $V_i(x_1,x_2) = \{y \in S \mid |y - x_i| < |y - x_j|\}$ and $V_0(x_1,x_2) = \{y \in S \mid |y - x_1| = |y - x_2|\}$.[4] We refer to f as a *demand function.*

This game is a symmetric game, i.e., the payoff functions satisfy

$$u_2(y,x) = u_1(x,y) \quad \forall x,y \in S, \tag{2}$$

which we will refer to as *player symmetry.* In addition, each u_i satisfies yet another symmetry, which we will call *location symmetry*:

$$u_i(x,y) = u_i(L-x, L-y) \quad \forall x,y \in S. \tag{3}$$

2.2 Potential games

There are several notions of potential functions (potentials, for short). Let $N = \{1,\dots,n\}$ and let $((S_i)_{i\in N}, (u_i)_{i\in N})$ be a general n-person game in strategic form, where S_i and $u_i\colon \bigtimes_{j\in N} S_j \to \mathbb{R}$ are the strategy set and the payoff function of player $i \in N$, respectively. For $i \in N$, $s \in \bigtimes_{j\in N} S_j$, and $s_i' \in S_i$, we denote by $s \setminus s_i' \in \bigtimes_{j\in N} S_j$ a strategy profile obtained from s by replacing the ith element with s_i'. Let $P\colon \bigtimes_{j\in N} S_j \to \mathbb{R}$. P is said to be a *generalized ordinal potential* of G (Monderer and Shapley, 1996) if for any i, s, s_i'

$$u_i(s \setminus s_i') > u_i(s) \implies P(s \setminus s_i') > P(s). \tag{4}$$

P is said to be a *best-response potential* of G (Voorneveld, 2000) if for any i, s

$$\arg\max_{s_i'} u_i(s \setminus s_i') = \arg\max_{s_i'} P(s \setminus s_i'). \tag{5}$$

If "$=$" in (5) is weakened to "$\supseteq$" then P is called a *pseudo-potential* of G (Dubey et al., 2006). P is called a *quasi-potential* of G (Schipper, 2004) if for any s

$$s_i \in \arg\max_{s_i' \in S_i} u_i(s \setminus s_i') \ \forall i \iff s \in \arg\max_{s' \in S} P(s'). \tag{6}$$

A game having a generalized ordinal potential is called a generalized ordinal potential game, and so on. If a game G belongs to any one of these classes of potential games, and if the potential therein has a maximum, then any maximizer of the potential is a pure Nash equilibrium of G.

A path is a nonempty finite or infinite sequence of strategy profiles $(s^1, s^2, \dots)$ such that every s^k and s^{k+1} differ in exactly one, say the $i(k)$th, coordinate. A finite path $(s^1, s^2, \dots, s^m)$ is called *cyclic* if $s^m = s^1$; *trivial* if $m = 1$. A path is an *improvement path* if $u_{i(k)}(s^{k+1}) > u_{i(k)}(s^k)$ for every k. It is known that there is no non-trivial cyclic improvement path in a generalized ordinal potential game (Monderer and Shapley, 1996).

[4] We borrowed the notations $V_i(x_1,x_2)$ of Prisner (2011).

3 Analysis

3.1 The results

Let $G = (S, f)$ and let $u_i \colon S^2 \to \mathbb{R}$ be defined by (1). As is well-known, the best response may be empty if the demand function f is constant ($f = c$): Indeed, since

$$u_1(x_1, x_2) = \begin{cases} \frac{x_1+x_2}{2}c & \text{if } x_1 < x_2 \\ \frac{L}{2}c & \text{if } x_1 = x_2 \\ (L - \frac{x_1+x_2}{2})c & \text{if } x_1 > x_2, \end{cases}$$

we have

$$\arg\max_{x_1 \in S} u_1(x_1, x_2) = \begin{cases} \emptyset & \text{if } x_2 \neq L/2 \\ \{L/2\} & \text{if } x_2 = L/2. \end{cases}$$

Clearly, this game does not admit continuous best-response potential (by Weierstrass's Theorem). As we shall see in Remark 3.1 below, the situation is not so different if f is strictly decreasing and $\frac{1}{2}f(0) < f(\frac{L}{2})$, a situation where the Principle of Minimum Differentiation holds true (Anderson et al., 1992, page 282).[5] Let us first consider the case $\frac{1}{2}f(0) \geq f(\frac{L}{2})$.

Let $a, b \in S$. We now let

$$\mathbf{L}(a) := \int_0^a f(z)dz, \quad \mathbf{R}(b) := \int_0^{L-b} f(z)dz, \quad \mathbf{M}(a, b) := 2\int_0^{\frac{|a-b|}{2}} f(z)dz,$$

and express the payoff functions given by (1) as

$$u_i(x_1, x_2) = \begin{cases} \mathbf{L}(x_i) + \frac{1}{2}\mathbf{M}(x_1, x_2) & \text{if } x_i < x_j \\ \mathbf{R}(x_i) + \frac{1}{2}\mathbf{M}(x_1, x_2) & \text{if } x_i > x_j \\ \frac{1}{2}\big(\mathbf{L}(x_i) + \mathbf{R}(x_i)\big) & \text{if } x_i = x_j. \end{cases}$$

Define $P \colon S^2 \to \mathbb{R}$ by

$$P(x_1, x_2) := \mathbf{L}(\min\{x_1, x_2\}) + \mathbf{R}(\max\{x_1, x_2\}) + \frac{1}{2}\mathbf{M}(x_1, x_2). \tag{7}$$

Note that P is continuous irrespective of the continuity of f.

Proposition 3.1. *$G = (S, f)$ with $S = [0, L]$ and a strictly decreasing f satisfying*

$$\frac{1}{2}f(0) \geq f(\frac{L}{2}) \tag{8}$$

admits a continuous best-response potential $P \colon S^2 \to \mathbb{R}$ defined by (7).

[5] To be precise, it holds true iff $\frac{1}{2}f(0) \leq f(\frac{L}{2})$ (Anderson et al. Eq.(8.6)). See also our Concluding Remark 2.

Proof. By player symmetry (2) and $P(x_1, x_2) = P(x_2, x_1)$, it suffices to show (5) for $i = 1$. Let

$$B(x_2) := \arg\max_{x_1 \in S} u_1(x_1, x_2) \quad \text{and} \quad M(x_2) := \arg\max_{x_1 \in S} P(x_1, x_2).$$

By location symmetry (3) and $P(x_1, x_2) = P(L - x_1, L - x_2)$, it suffices to show $B(x_2) = M(x_2)$ for $x_2 \leq \frac{L}{2}$. We distinguish two cases.

Case $x_2 < \frac{L}{2}$: Note that $\frac{1}{2}f(0) - f(L - x_2) > \frac{1}{2}f(0) - f(\frac{L}{2}) \geq 0$, i.e., $\frac{1}{2}f(0) > f(L - x_2)$. By the continuity of f, we can choose $\epsilon > 0$ such that $\epsilon < L - x_2 - \epsilon$ and $\frac{1}{2}f(\epsilon) > f(L - x_2 - \epsilon)$. If $x_1 = x_2$ then

$$\begin{aligned} u_1(x_1 + \epsilon, x_2) - u_1(x_1, x_2) &= \frac{1}{2}\mathbf{M}(x_2, x_2 + \epsilon) + \mathbf{R}(x_2 + \epsilon) - \frac{1}{2}(\mathbf{R}(x_2) + \mathbf{L}(x_2)) \\ > P(x_1 + \epsilon, x_2) - P(x_1, x_2) &= \frac{1}{2}\mathbf{M}(x_2, x_2 + \epsilon) + \mathbf{R}(x_2 + \epsilon) - \mathbf{R}(x_2) \\ &= \int_0^{\frac{\epsilon}{2}} f(z)dz - \int_{L - x_2 - \epsilon}^{L - x_2} f(z)dz > \int_0^{\epsilon} \frac{1}{2} f(z)dz - \int_{L - x_2 - \epsilon}^{L - x_2} f(z)dz > 0, \end{aligned}$$

where the first inequality is by $\mathbf{L}(x_2) < \mathbf{R}(x_2)$, and the second by $2\int_0^{\frac{\epsilon}{2}} f(z)dz > \int_0^{\epsilon} f(z)dz$ (due to the strict decreasingness). Thus $x_1 \neq x_2$ if $x_1 \in B(x_2) \cup M(x_2)$. Note that, if $x_2 > 0$, then for $x_1' = x_2 - h$ and $x_1'' = x_2 + h$ such that $0 \leq x_1' < x_2 < x_1''$,

$$u_1(x_1', x_2) < u_1(x_1'', x_2) \text{ and } P(x_1', x_2) < P(x_1'', x_2), \tag{9}$$

since x_1'' will add (resp. increase) demand from the interval $[L - 2x_2, L]$ for $u_1(\cdot, x_2)$ (resp. $P(\cdot, x_2)$), compared to x_1'. Thus $B(x_2) = \arg\max_{x_1 > x_2}\big(\mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$ and $M(x_2) = \arg\max_{x_1 > x_2}\big(\mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$. Since $\mathbf{L}(x_2)$ is a constant, we have $B(x_2) = M(x_2)$.

Case $x_2 = \frac{L}{2}$: In this case, $B(x_2)$ and $M(x_2)$ are symmetric in that $y \in B(x_2) \iff L - y \in B(x_2)$ and $y \in M(x_2) \iff L - y \in M(x_2)$. Also, both $u_1(\cdot, x_2)$ and $P(\cdot, x_2)$ are continuous. Hence $B(x_2) \cap [x_2, L] \neq \emptyset$ and $M(x_2) \cap [x_2, L] \neq \emptyset$. We are done if $B(x_2) \cap [x_2, L] = M(x_2) \cap [x_2, L]$ is shown. Now, as $x_2 = \frac{L}{2}$, we have $u_1(x_1, x_2) = \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)$ for any $x_1 \in [x_2, L]$. Observe that $B(x_2) \cap [x_2, L] = \arg\max_{x_1 \geq x_2}\big(\mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$ and $M(x_2) \cap [x_2, L] = \arg\max_{x_1 \geq x_2}\big(\mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)\big)$, which, by the constancy of $\mathbf{L}(x_2)$, imply $B(x_2) \cap [x_2, L] = M(x_2) \cap [x_2, L]$. $\square$

Remark 3.1. Let $x_2 < \frac{L}{2}$. Then, with $x_1 = x_2$,

$$\begin{aligned} u_1(x_1, x_2) &= \frac{1}{2}\big(\mathbf{L}(x_2) + \mathbf{R}(x_2)\big) \\ &< \mathbf{R}(x_2) = \lim_{y_1 \downarrow x_1}\big(\mathbf{R}(y_1) + \frac{1}{2}\mathbf{M}(x_2, y_1)\big) = \lim_{y_1 \downarrow x_1} u_1(y_1, x_2). \end{aligned} \tag{10}$$

That is, $u_1(\cdot, x_2)$ is not continuous at $x_1 = x_2$, and $u_1(x_1, x_2) < u_1(x_1+\epsilon, x_2)$ for a small $\epsilon > 0$. Nevertheless, the condition $\frac{1}{2}f(0) \geq f(\frac{L}{2})$ ensures that $B(x_2) \neq \emptyset$ for every $x_2 \in [0, L]$ as Proposition 3.1 suggests (in fact $B(x_2) = M(x_2) \neq \emptyset$ for every $x_2 \in [0, L]$). If $\frac{1}{2}f(0) < f(\frac{L}{2})$, on the other hand, $B(x_2)$ may be empty for some x_2. To see this, observe that for $y_1 > x_2$ with $x_2 < \frac{L}{2}$, $u_1(\cdot, x_2)$ is continuously differentiable at y_1 and

$$\begin{aligned}\frac{\partial}{\partial y_1}u_1(y_1, x_2) &= \frac{\partial}{\partial y_1}\left(\mathbf{R}(y_1) + \frac{1}{2}\mathbf{M}(x_2, y_1)\right)\\ &= \frac{\partial}{\partial y_1}\left(\int_0^{L-y_1} f(z)dz + \int_0^{\frac{y_1-x_2}{2}} f(z)dz\right) = -f(L-y_1) + \frac{1}{2}f(\frac{y_1-x_2}{2}). \quad (11)\end{aligned}$$

If $\frac{1}{2}f(0) < f(\frac{L}{2})$, and if $x_2 < \frac{L}{2}$ is sufficiently close to $\frac{L}{2}$, then, by the continuity of f, (11) converges to $-f(L-x_2) + \frac{1}{2}f(0) < 0$ as $y_1 \downarrow x_2$. That is, we have $\frac{\partial}{\partial x_1}u_1(x_1, x_2) < 0$ for all $x_1 > x_2$, since $-f(L-x_1) + \frac{1}{2}f(\frac{x_1-x_2}{2})$ is decreasing in x_1. With (10), this says that $u_1(\cdot, x_2)$ has no maximum on $[x_2, L]$. Also, with (9), which also holds here, it has no maximum on $[0, L]$, namely, $B(x_2) = \emptyset$, while $M(x_2) \neq \emptyset$ since P is continuous. Hence in this case P defined by (7) cannot be a best-response potential, nor even a pseudo-potential since $B(x_2) \not\supseteq M(x_2)$. ∎

Before we proceed, we note that if (x_1, x_2) is an equilibrium of G, then $x_1 + x_2 = L$, and the equilibrium is unique up to player symmetry.[6] To see this, note first that $(\frac{L}{2}, \frac{L}{2})$ is the unique equilibrium of G if f is constant. Asume then that f is strictly decreasing, $x_2 \leq x_1$, without loss of generality, and suppose $x_1 + x_2 < L$. If $x_2 = x_1$, then, by (10), we have $u_1(x_1, x_2) < u_1(x_1 + \epsilon, x_2)$ for a small $\epsilon > 0$, a contradiction. If $x_2 < x_1$, note that

$$\frac{\partial}{\partial x_1}u_1(x_1, x_2) = -f(L-x_1) + \frac{1}{2}f(\frac{x_1-x_2}{2}), \ \frac{\partial}{\partial x_2}u_2(x_1, x_2) = f(x_2) - \frac{1}{2}f(\frac{x_1-x_2}{2}),$$

and $x_2 \neq 0$ since $\frac{\partial}{\partial x_2}u_2(x_1, 0) > 0$. Then $\frac{\partial}{\partial x_1}u_1(x_1, x_2) = \frac{\partial}{\partial x_2}u_2(x_1, x_2) = 0$ by the first order condition, and $f(L-x_1) = f(x_2)$, contradicting the strict decreasingness of f. Hence $x_1 + x_2 \geq L$. Note that $(L-x_1, L-x_2)$ is also an equilibrium by location symmetry (3), and the above argument also implies $(L-x_1) + (L-x_2) \geq L$. Hence $x_1 + x_2 = L$. For the uniqueness, suppose that (x_1, x_2) and (x_1', x_2') are two equilibria such that $x_2 \leq x_1$, $x_2' \leq x_1'$, and $x_1' < x_1$. Then $u_1(x_1, x_2) \geq u_1(x_1', x_2)$ implies $\int_0^{\frac{x_1-x_2}{2}} f(z)dz + \int_0^{L-x_1} f(z)dz \geq \int_0^{\frac{x_1'-x_2}{2}} f(z)dz + \int_0^{L-x_1'} f(z)dz$, and $u_1(x_1, x_2') \leq u_1(x_1', x_2')$ implies $\int_0^{\frac{x_1-x_2'}{2}} f(z)dz + \int_0^{L-x_1} f(z)dz \leq \int_0^{\frac{x_1'-x_2'}{2}} f(z)dz + \int_0^{L-x_1'} f(z)dz$. Subtracting,

$$\int_{\frac{x_1-x_2'}{2}}^{\frac{x_1-x_2}{2}} f(z)dz \geq \int_{\frac{x_1'-x_2'}{2}}^{\frac{x_1'-x_2}{2}} f(z)dz.$$

[6] These points are also shown by Anderson et al. (1992). However, we include our proofs for the sake of completeness.

Since $\frac{x_1-x_2'}{2} > \frac{x_1'-x_2'}{2}$ and $\frac{x_1-x_2}{2} - \frac{x_1-x_2'}{2} = \frac{x_1'-x_2}{2} - \frac{x_1'-x_2'}{2}$, this contradicts the strict decreasingness of f. Hence equilibrium must be unique; it is unique up to player symmetry since $\{(x_1, x_2), (x_2, x_1), (L - x_1, L - x_2), (L - x_2, L - x_1)\} = \{(x_1, x_2), (x_2, x_1)\}$ by $x_1 + x_2 = L$.

Proposition 3.2. *$G = (S, f)$ with a strictly decreasing f admits a continuous quasi-potential $P\colon S^2 \to \mathbb{R}$ defined by* (7)*.*

Proof. We show '$\Longleftarrow$' in (6), i.e., that any maximizer of P is an equilibrium. This and the uniqueness of equilibrium (up to player symmetry) imply (6).

Suppose $(x_1, x_2) \in \arg\max_{s\in S^2} P(s)$. Assume $x_2 \leq x_1$ without loss of generality (by $P(x_1, x_2) = P(x_2, x_1)$). Note that if $x_1 + x_2 < L$, then for $\epsilon > 0$ such that $x_2 + \epsilon < L - x_1 - \epsilon$,

$$\begin{aligned} P(x_1+\epsilon, x_2+\epsilon) - P(x_1, x_2) &= \big(\mathbf{L}(x_2+\epsilon) - \mathbf{L}(x_2)\big) + \big(\mathbf{R}(x_1+\epsilon) - \mathbf{R}(x_1)\big) \\ &= \int_{x_2}^{x_2+\epsilon} f(z)dz - \int_{L-x_1-\epsilon}^{L-x_1} f(z)dz > 0. \end{aligned}$$

Likewise, if $x_1 + x_2 > L$, then for $\epsilon > 0$ such that $x_2 - \epsilon > L - x_1 + \epsilon$, $P(x_1 - \epsilon, x_2 - \epsilon) > P(x_1, x_2)$. Since these contradict the maximality of $P(x_1, x_2)$, we must have $x_1 + x_2 = L$. Then $\mathbf{L}(x_2) = \mathbf{R}(x_1)$. We distinguish two cases.

Case $x_2 = \frac{L}{2}$: In this case that $P(x_1, x_2) \geq P(x_1', x_2)$ for any $x_1' \in S$ implies that $\mathbf{R}(x_1) \geq \mathbf{R}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$ for any $x_1' \geq x_2$ and $\mathbf{L}(x_1) \geq \mathbf{L}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$ for any $x_1' \leq x_2$. That is, $u_1(x_1, x_2) \geq u_1(x_1', x_2)$ for any $x_1' \geq x_2$ and any $x_1' \leq x_2$, respectively (note that $\mathbf{R}(x_1) = \mathbf{L}(x_1) = \frac{1}{2}(\mathbf{L}(x_1) + \mathbf{R}(x_1))$). Hence $x_1 \in \arg\max_{x_1'\in S} u_1(x_1', x_2)$. We also have $x_2 \in \arg\max_{x_2'\in S} u_2(x_1, x_2')$ by player symmetry.

Case $x_2 < \frac{L}{2}$: (a) If $x_1' > x_2$, then $P(x_1', x_2) = \mathbf{L}(x_2) + \mathbf{R}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$ and $u_1(x_1', x_2) = \mathbf{R}(x_1') + \frac{1}{2}\mathbf{M}(x_1', x_2)$, so $P(x_1, x_2) \geq P(x_1', x_2)$ implies $u_1(x_1, x_2) \geq u_1(x_1', x_2)$. (b) If $x_1' < x_2$, recall that there is $x_1'' > x_2$ such that $u_1(x_1'', x_2) > u_1(x_1', x_2)$ (see (9)). Since $u_1(x_1, x_2) \geq u_1(x_1'', x_2)$ by (a), we have $u_1(x_1, x_2) \geq u_1(x_1', x_2)$. (c) If $x_1' = x_2$, then $P(x_1', x_2) \leq P(x_1, x_2)$ reads as $\mathbf{L}(x_2) + \mathbf{R}(x_1') \leq \mathbf{L}(x_2) + \mathbf{R}(x_1) + \frac{1}{2}\mathbf{M}(x_2, x_1)$, so $\mathbf{L}(x_2) + \mathbf{R}(x_1') \leq \mathbf{L}(x_2) + \mathbf{R}(x_1) + \mathbf{M}(x_2, x_1)$. Noting that $u_1(x_1', x_2) = u_2(x_1', x_2)$ by $x_1' = x_2$, and $u_1(x_1, x_2) = u_2(x_1, x_2)$ by $u_1(x_1, x_2) = u_2(x_2, x_1) = u_2(L - x_2, L - x_1) = u_2(x_1, x_2)$, this says that $2u_1(x_1', x_2) \leq 2u_1(x_1, x_2)$. Hence $x_1 \in \arg\max_{x_1'\in S} u_1(x_1', x_2)$. We also have $x_2 \in \arg\max_{x_2'\in S} u_2(x_1, x_2')$ by player symmetry. □

In passing, we note that $G = (S, f)$ with $S = [0, L]$ and a constant f has a unique equilibrium $(\frac{L}{2}, \frac{L}{2})$, as is well known. Clearly, $\bar{P}\colon S^2 \to \mathbb{R}$ defined by

$$\bar{P}(x_1, x_2) := -\left(\left|\frac{L}{2} - x_1\right| + \left|\frac{L}{2} - x_2\right|\right), \tag{12}$$

for example, is a continuous quasi-potential of G.

3.2 Non-existence of a generalized ordinal potential

Having established that G belongs to the class of quasi-potential games, and more strongly best-response potential games (with a continuous best-response potential) if f is a sufficiently decreasing strictly decreasing function, we now show that G is not necessarily a generalized ordinal potential game, to narrow the class to which G belongs.

Proposition 3.3. *$G = (S, f)$ with a sufficiently decreasing strictly decreasing f is not necessarily a generalized ordinal potential game.*

Proof. If G has a generalized ordinal potential then it cannot have any non-trivial cyclic improvement path. We provide a counterexample, i.e., an example of G having a non-trivial cyclic improvement path. Let $f(z) = w^z$ with a constant w such that $0 < w \leq 1$. Then, for the game $G = (S, f)$ with $S = [0, 3]$, player 1's payoffs at integer points are as given in Figure 1. By Proposition 3.1, this game is a best-response potential game if $\frac{1}{2}f(0) \geq f(\frac{3}{2})$, i.e., if $w \leq (\frac{1}{2})^{\frac{2}{3}}$. The rounded payoff values when $w = \frac{1}{2}$ are as shown in Figure 1. As we can see, we have a cyclic improvement path $(2, 0) \to (1, 0) \to (1, 3) \to (2, 3) \to (2, 0)$. $\square$

	0	1	2	3
0	$\frac{w^3-1}{2\ln w}$	$\frac{w^{\frac{1}{2}}-1}{\ln w}$	$\frac{w-1}{\ln w}$	$\frac{w^{\frac{3}{2}}-1}{\ln w}$
1	$\frac{w^2+w^{\frac{1}{2}}-2}{\ln w}$	$\frac{w^2+w-2}{2\ln w}$	$\frac{w+w^{\frac{1}{2}}-2}{\ln w}$	$\frac{2w-2}{\ln w}$
2	$\frac{2w-2}{\ln w}$	$\frac{w+w^{\frac{1}{2}}-2}{2\ln w}$	$\frac{w^2+w-2}{2\ln w}$	$\frac{w^2+w^{\frac{1}{2}}-2}{\ln w}$
3	$\frac{w^{\frac{3}{2}}-1}{\ln w}$	$\frac{w-1}{\ln w}$	$\frac{w^{\frac{1}{2}}-1}{\ln w}$	$\frac{w^3-1}{2\ln w}$

(a)

	0	1	2	3
0	0.631	0.423	0.721	0.933
1	1.505	0.902	1.144	1.443
2	1.443	1.144	0.902	1.505
3	0.933	0.721	0.423	0.631

(b)

Figure 1: A game having a non-trivial cyclic improvement path (player 1's payoff when $S = [0, 3]$ and $w = \frac{1}{2}$).

4 Concluding remarks

1. Let $G = (S, f)$ and $S = [0, L]$. We have shown that if f is strictly decreasing and satisfies $\frac{1}{2}f(0) \geq f(\frac{L}{2})$ then G admits a continuous best-response potential (Proposition 3.1) but may not be a generalized ordinal potential game (Proposition 3.3). The existence of a pure Nash equilibrium follows from the continuity of the best-response potential P defined by (7). In general, the same continuous function P is a quasi-potential if f is a strictly decreasing function (Proposition 3.2), and such a G has a unique equilibrium up to player symmetry

(see fn. 6). Clearly, G with a constant f also has a continuous quasi-potential and a unique equilibrium $(\frac{L}{2}, \frac{L}{2})$.

2. Let $G = (S, f)$ with a strictly decreasing f. Then we can locate the unique (up to player symmetry) equilibrium (x_1, x_2) of G in the following way.[7] Suppose $x_2 \leq \frac{L}{2}$. Here $x_2 \neq 0$, since otherwise $(x_1, x_2) = (L, 0)$, and $u_1(\epsilon, 0) > \int_\epsilon^L f(z)dz > \int_0^{\frac{L}{2}} f(z)dz = u_1(L, 0)$ for a sufficiently small $\epsilon > 0$, a contradiction. Note that for (x_1, x_2) with $x_2 \in]0, \frac{L}{2}[$ (and $x_1 \in]\frac{L}{2}, L[$), we have $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = -f(L - x_1) + \frac{1}{2} f(\frac{x_1 - x_2}{2})$ as in (11), so $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = -f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2)$ by $x_1 + x_2 = L$. Also $-f(0) + \frac{1}{2} f(\frac{L}{2}) < 0$ by the strict decreasingness of f, and $-f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2)$ is strictly increasing in x_2 over $]0, \frac{L}{2}[$ due to the strict decreasingness of f. Therefore, if $-f(\frac{L}{2}) + \frac{1}{2} f(0) > 0$, then (x_1, x_2) is found by solving

$$-f(x_2) + \frac{1}{2} f(\frac{L}{2} - x_2) = 0, \tag{13}$$

with $x_1 = L - x_2$. Eq. (13) is the first order condition $\frac{\partial}{\partial x_1} u_1(x_1, x_2) = 0$ that has to be satisfied at the equilibrium (x_1, x_2) such that $x_2 < x_1$. If $-f(\frac{L}{2}) + \frac{1}{2} f(0) \leq 0$, then (13) fails at every $x_2 \in]0, \frac{L}{2}[$, and recalling that $x_2 \neq 0$, we must have $(x_1, x_2) = (\frac{L}{2}, \frac{L}{2})$. The last condition $\frac{1}{2} f(0) \leq f(\frac{L}{2})$ is the necessary and sufficient condition for the Principle of Minimum Differential to hold (Anderson et al., 1992). Thus, with Proposition 3.1, $G = (S, f)$ such that $\frac{1}{2} f(0) = f(\frac{L}{2})$ is a game that admits a continuous best-response potential and the Principle.

3. We have shown that any $G = (S, f)$ in this paper has some potential function, among which the quasi-potential function is the most general one. A further generalization of potential function is possible: replace '$\iff$' in (6) with '$\Longleftarrow$'. Such a potential function may be called a *weak quasi-potential function*. Note that the continuity of f is not used in the part of the proof of Proposition 3.2, where '$\Longleftarrow$' in (6) is being proved. Thus, $G = (S, f)$ with a strictly decreasing not necessarily continuous f can be said to be a weak quasi-potential game.

4. As a final remark, we note that: *If the strategy set is circular (as in Salop (1979)), then the two-person location games with payoff functions given by* (1) *is an exact potential game.* The proof is straightforward. The game is then an identical interest game, which is an exact potential game (Monderer and Shapley, 1996).

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[7] The characterization of equilibrium below is also obtained in Anderson et al. (1992). Again, we include our short proof here for the sake of completeness.

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